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GENERALIZED ANALYSIS OF SHEAR DEFORMABLE RINGS AND CURVED BEAMS

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Abstract—An accurate shear deformable theory for the analysis of the complete dynamic response of curved beams of constant curvature is presented. The equations presented here are very general in the sense that any problem of curved beam and circular rings can be addressed. It is indicated that the classical thin beam theory and the Timoshenko-type shear deformation theory are obtainable from the present theory as special cases. The present formulation accounts for shear deformation and rotary inertia and hence is applicable to the analysis of thick curved beams and rings. The theory assumes parabolic variation for shear strains (hence obviates the use of shear correction factors as usually done in the case of Timoshenko-type theory) and involves six displacement parameters of the centre-line—three translations and three rotations. It is pointed out that for certain composite curved beams and rings the in-plane and out-of-plane vibrations are coupled. In such cases complete analysis, rather than separate in-plane and out-of-plane analyses, is required. The numerical results presented illustrate the effect of coupling on various vibrational frequencies.

INTRODUCTION

The analysis of circular ring has been a topic of interest to research workers for over a century. The earliest work is usually attributed to Hoppe[1]. This topic is still of concern at the present time because ring elements are important components in many modern structures. Also the increasing use of fibre-reinforced composite materials in ring-type structures demands an accurate description of the mathematical model to analyse such structures. Many authors have contributed to the improvement of our knowledge on the behaviour of rings and have systematically investigated the effects of rotary inertia, shear deformation and centre-line extension on their natural vibration[2–13]. The discussion on very early developments on the analysis of rings may be found in the work of Gardner and Bert[6].

It is evident from the literature that there are three categories of ring vibration problems namely: purely in-plane vibrations, purely out-of-plane vibrations and coupled in-plane and out-of-plane vibration problems. Among others, Ambati *et al.*[2], Davis *et al.*[3], Kirkhope[4, 5], and Gardner and Bert[6] have studied the in-plane vibration of rings using the theories of varied accuracy. Ambati *et al.*[2] have presented an elasticity solution to the in-plane vibration problem and have tabulated extensive numerical results. Kirkhope[4, 5] solved the same problem using the shear deformation theory similar to Timoshenko's straight beam theory. Gardner and Bert[6] used the more realistic shear deformation theory, by extending Levinson's[14] straight beam theory, to analyse the in-plane vibration problem and also they have presented useful experimental results.

The out-of-plane vibration problem has been studied by Rao[7], Kirkhope[8], and Bickford and Maganty[9] using Timoshenko-type shear deformation theory. The formulations of Rao[7] and Kirkhope[8] do not account for the variations in curvature across the cross-section. This deficiency has been corrected by Bickford and Maganty[9]. Thus, among the results presented so far, on the out-of-plane vibration problem, those of Bickford and Maganty[9] seem to be the most accurate.

When the cross-sectional shape of the ring is unsymmetrical the vibrations can no longer consist of purely in-plane or purely out-of-plane motions. Instead each free vibration is composed of a combination of both. This is so because principal axes of inertia do not lie in the ring plane. Thus leading to the third category of ring vibration problems. It is interesting to note that, as will be demonstrated later, the coupled in-plane and out-of-plane motions may occur when the cross-section of the ring is made up of laminated

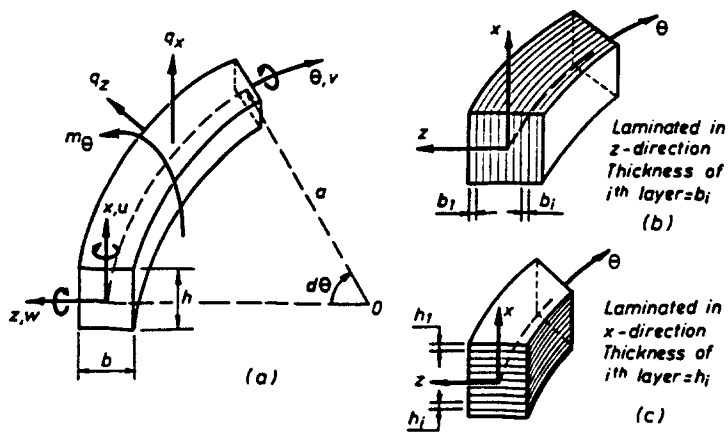

Fig. 1. Dimensions and coordinate system.

construction; despite being symmetrical and despite having principal axes of inertia lying in the ring plane. This is so because in the laminated cross-section coupling arises mainly due to the orientation of individual laminates which make up the ring cross-section.

It is surprising to note that there are hardly any references on the generalized (i.e. third category) analysis of the rings except the works of Endo[10], Hawkings[11] and Kirkhope *et al.*[12]. All these works deal with unsymmetrical cross-sections made up of homogeneous materials. To the author's knowledge the coupled vibration problem of rings made up of laminated cross-sections has not been dealt with in the literature. Thus, it is the purpose of this paper to give an accurate mathematical model to analyse the generalized vibration problem of rings made of laminated construction.

Before proceeding with the present formulation, some pertinent points regarding the same are in order. In the present theory shear strain variations are assumed to be parabolic (distorted parabola to be more specific) across the cross-section, unlike constant variation as assumed in the Timoshenko-type theories. This formulation is based on the author's earlier work on plates and shells[15, 16]. The features of the present theory (namely parabolic shear strain variation and vanishing shear strain at the intrados and extrados) are similar to those of Gardner and Bert[6] for the case of in-plane vibrations of rings. However, the essential and important difference in the present theory and that of Ref. [6] is that the classical thin ring and the Timoshenko-type shear deformation theories can be obtained as special cases from the present theory whereas, it is not so with that of Ref. [6].

THEORETICAL DEVELOPMENT

Figure 1 shows the coordinate system and the dimensions of the curved element and the lamination scheme (a list of nomenclature is given in the Appendix). The present theory has been formulated by completely abandoning the hypothesis of the classical thin ring theory that the plane sections remain plane and normal. As in most of the displacement based formulations we start with the following assumed displacement components:

$$u(x,\theta,z,t) = u_0 - z\phi$$

$$v(x,\theta,z,t) = \left(\frac{a+z}{a}\right)(v_0 + f_z v_1 + f_x u_1 - x u_0') - z w_0'$$

$$w(x,\theta,z,t) = w_0 + x\phi. \qquad (1)$$

Here $(\,)' = \partial(\,)/a\,\partial\theta$ and f_z, f_x and their derivatives are given by

$$f_z = z\left(1-\frac{4z^2}{3b^2}\right);\quad f_x = x\left(1-\frac{4x^2}{3h^2}\right) \tag{2}$$

$$f_z^* = \frac{df_z}{dz} = \left(1-\frac{4z^2}{b^2}\right);\quad f_x^* = \frac{df_x}{dx} = \left(1-\frac{4x^2}{h^2}\right). \tag{3}$$

The preceding kinematical assumptions allow non-uniform shearing of the cross-sections. It may be said here that u_1 and v_1 are the flexural rotations, in addition to the usual u_0' and w_0' terms, and ϕ is the torsional rotation. The strain–displacement relations in the cylindrical coordinate system are written as[17]

$$\varepsilon_x = \frac{\partial u}{\partial x};\quad \varepsilon_\theta = \left(\frac{a}{a+z}\right)\left(\frac{\partial v}{a\,\partial\theta}+\frac{w}{a}\right);\quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{\theta x} = \frac{\partial v}{\partial x}+\left(\frac{a}{a+z}\right)\frac{\partial u}{a\,\partial\theta};\quad \gamma_{xz} = \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

$$\gamma_{\theta z} = \left(\frac{a}{a+z}\right)\left(\frac{\partial w}{a\,\partial\theta}-\frac{v}{a}\right)+\frac{\partial v}{\partial z}. \tag{4}$$

These are the exact relations unlike the first approximation-type as adopted by Gardner and Bert[6]. Using eqns (1) in eqns (4) we obtain the following strain components relevant to the present problem:

$$\varepsilon_\theta = v_0'+f_z v_1'+f_x u_1'-xu_0''-z\left(\frac{a}{a+z}\right)w_0''+\left(\frac{a}{a+z}\right)\frac{w_0}{a}+\left(\frac{a}{a+z}\right)\frac{x}{a}\phi$$

$$\gamma_{\theta x} = \left(\frac{a+z}{a}\right)f_x^* u_1-\frac{z}{a}\left(\frac{2a+z}{a+z}\right)u_0'-\left(\frac{a}{a+z}\right)z\phi'$$

$$\gamma_{\theta z} = \left(\frac{a+z}{a}\right)f_z^* v_1+\left(\frac{a}{a+z}\right)x\phi'. \tag{5}$$

At this stage it is worth commenting on some salient features of the present formulation. In the case of purely in-plane motions ($u_0 = u_1 = \phi = \gamma_{\theta x} = 0$) shear strain ($\gamma_{\theta z}$) varies parabolically and is zero at the intrados and the extrados. In the case of out-of-plane motions ($w_0 = v_1 = 0$) shear strain ($\gamma_{\theta x}$) varies parabolically and the most important feature of coupling between bending and torsion is retained.

However, in the case of out-of-plane motions or coupled in-plane and out-of-plane motions shear strains do not vanish at the free surfaces though the non-linear variation is retained. In the author's opinion, violation of shear free surface conditions has no real significance on the numerical results as long as the non-linear variation for shear strains is maintained. Moreover, it is an extremely difficult task to formulate a general theory for rings satisfying the shear-free conditions without going into complex mathematical manipulations which would definitely defeat the very purpose of one-dimensional simplified analysis.

Finally, some remarks on the forms of f_z (and f_x) are in order. Almost any function the first derivative of which vanishes at $\pm b/2$ (and $\pm h/2$) and is non-zero elsewhere can be chosen. Some such functions, in addition to those given in eqns (2), can be found in Ref. [16]. If we use $f_z = z$ (and $f_x = x$) we obtain Timoshenko-type shear deformation theory and if we use $f_z = f_x = 0$ we obtain the classical thin ring theory. Alternately, we can ignore the terms associated with u_1 and v_1 displacement parameters to obtain the classical thin ring theory. Since the non-linear variation has been used for shear strain descriptions across the cross-section there arises no need to introduce shear correction factors, as usually done in Timoshenko-type shear deformation theory.

The constitutive relations for the ring material, or for any layer in the case of laminates, are assumed as

$$\sigma_\theta = c_{11}\varepsilon_\theta;\quad \tau_{\theta x} = c_{66}\gamma_{\theta x};\quad \tau_{\theta z} = c_{44}\gamma_{\theta z}. \tag{6}$$

Strain energy, kinetic energy and the work potential of the external loads, respectively, may be written as

$$U = \iiiint_{tzx\theta} (c_{11}\varepsilon_\theta^2 + c_{66}\gamma_{\theta x}^2 + c_{44}\gamma_{\theta z}^2)\left(\frac{a+z}{a}\right)a\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \tag{7}$$

$$K = \iiiint_{tzx\theta} \rho(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\left(\frac{a+z}{a}\right)a\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t \tag{8}$$

$$W = \iint_{t\theta} (q_x u_0 + q_z w_0 + m_\theta \phi)a\,\mathrm{d}\theta\,\mathrm{d}t. \tag{9}$$

In the above an "overdot" indicates differentiation with respect to the time coordinate. The generalized stress resultants have the following definitions:

$$(N_\theta^*, M_z^*, M_x) = \iint \sigma_\theta(1, x, z)\,\mathrm{d}x\,\mathrm{d}z$$

$$(N_\theta, M_z, \bar{M}_z, \bar{M}_x) = \iint \sigma_\theta(1, x, f_x, f_z)\left(\frac{a+z}{a}\right)\mathrm{d}x\,\mathrm{d}z$$

$$(\bar{Q}_x, \bar{Q}_z) = \iint (\tau_{\theta x} f_x^*, \tau_{\theta z} f_z^*)\left(\frac{a+z}{a}\right)^2\mathrm{d}x\,\mathrm{d}z$$

$$(M_{\theta x}, M_{\theta z}) = \iint (-\tau_{\theta x} z, \tau_{\theta z} x)\,\mathrm{d}x\,\mathrm{d}z$$

$$Q_z = M_x' - \iint \rho \ddot{v} z\left(\frac{a+z}{a}\right)\mathrm{d}x\,\mathrm{d}z$$

$$Q_x = M_z' + \frac{M_{\theta x}}{a} - \iint \rho \ddot{v} x\left(\frac{a+z}{a}\right)^2\mathrm{d}x\,\mathrm{d}z$$

$$M_\theta = (M_{\theta x} + M_{\theta z}). \tag{10}$$

Using Hamilton's principle we obtain the following equilibrium equations in terms of the generalized forces:

$$N_\theta' = \iint \rho \ddot{v}\left(\frac{a+z}{a}\right)^2\mathrm{d}x\,\mathrm{d}z$$

$$Q_z' - \frac{1}{a}N_\theta^* = -q_z + \iint \rho \ddot{w}\left(\frac{a+z}{a}\right)\mathrm{d}x\,\mathrm{d}z$$

$$\bar{M}_x' - \bar{Q}_z = \iint \rho \ddot{v} f_z\left(\frac{a+z}{a}\right)^2\mathrm{d}x\,\mathrm{d}z$$

$$Q_x' = -q_x + \iint \rho \ddot{u}\left(\frac{a+z}{a}\right)\mathrm{d}x\,\mathrm{d}z$$

$$\bar{M}'_z - \bar{Q}_x = \iint \rho \ddot{v} f_x \left(\frac{a+z}{a}\right)^2 \mathrm{d}x\,\mathrm{d}z$$

$$\frac{1}{a} M_z^* - M_\theta = m_\theta + \iint \rho(z\ddot{u} - x\ddot{w}) \left(\frac{a+z}{a}\right) \mathrm{d}x\,\mathrm{d}z. \tag{11}$$

The boundary conditions require that one member of each of the following eight pairs or eight linearly independent combinations of them must be specified at $\theta = \text{const}$.

$$N_\theta, v_0;\quad Q_x, u_0;\quad Q_z, w_0;\quad M_z, u'_0;\quad M_x, w'_0;\quad \bar{M}_x, v_1;\quad \bar{M}_z, u_1;\quad M_\theta, \phi.$$

Using the strain–displacement relations (5) in eqns (10) and carrying out the indicated integration across the cross-section we obtain the required relations for generalized stress resultants in terms of displacement parameters as given in Appendix A. Substituting the stress resultants in eqns (11) we obtain six equilibrium equations in terms of six displacement parameters of the problem as

$$D_{11}v''_0 + D_{16}\frac{w'_0}{a} - D_{14}w'''_0 + D_{15}v''_1 - D_{12}u'''_0 + D_{13}u''_1 + D_{17}\frac{\phi'}{a}$$
$$= \rho_{11}\ddot{v}_0 - \rho_{15}\ddot{w}'_0 + \rho_{12}\ddot{v}_1 - \rho_{14}\ddot{u}'_0 + \rho_{13}\ddot{u}_1 \tag{12a}$$

$$D_{16}\frac{v'_0}{a} - D_{14}v'''_0 + D_{44}w_0^{\mathrm{IV}} - 2D_{46}\frac{w''_0}{a} + D_{66}\frac{w_0}{a^2} + D_{56}\frac{v'_1}{a} - D_{45}v'''_1$$
$$+ D_{24}u_0^{\mathrm{IV}} - D_{26}\frac{u''_0}{a} + D_{36}\frac{u'_1}{a} - D_{34}u'''_1 + D_{67}\frac{\phi}{a^2} - D_{47}\frac{\phi''}{a}$$
$$= q_z - \rho_{15}\ddot{v}'_0 - \bar{\bar{\rho}}_{11}\ddot{w}_0 + \rho_{55}\ddot{w}''_0 + \rho_{45}\ddot{u}''_0 - \rho_{25}\ddot{v}'_1 - \rho_{35}\ddot{u}'_1 - \bar{\bar{\rho}}_{12}\ddot{\phi} \tag{12b}$$

$$D_{15}v''_0 + D_{56}\frac{w'_0}{a} - D_{45}w'''_0 + D_{55}v''_1 - \bar{G}_{11}v_1 - D_{25}u'''_0 + D_{35}u''_1$$
$$+ \left(\frac{D_{57}}{a} - \bar{G}_{12}\right)\phi' = \rho_{12}\ddot{v}_0 - \rho_{25}\ddot{w}'_0 + \rho_{22}\ddot{v}_1 - \rho_{24}\ddot{u}'_0 + \rho_{23}\ddot{u}_1 \tag{12c}$$

$$-D_{12}v'''_0 + D_{24}w_0^{\mathrm{IV}} - D_{26}\frac{w''_0}{a} - D_{25}v'''_1 + D_{22}u_0^{\mathrm{IV}} - G_{22}\frac{u''_0}{a^2}$$
$$+ G_{12}\frac{u'_1}{a} - D_{23}u'''_1 - \left(\frac{D_{27}}{a} + \frac{G_{23}}{a}\right)\phi''$$
$$= q_x - \rho_{14}\ddot{v}'_0 + \rho_{45}\ddot{w}''_0 - \rho_{24}\ddot{v}'_1 - \rho_{11}\ddot{u}_0 + \rho_{44}\ddot{u}''_0 - \rho_{34}\ddot{u}'_1 + \bar{\rho}_{12}\ddot{\phi} \tag{12d}$$

$$D_{13}v''_0 + \frac{D_{36}}{a}w'_0 - D_{34}w'''_0 + D_{35}v''_1 + G_{12}\frac{u'_0}{a} - D_{23}u'''_0 + D_{33}u''_1 - G_{11}u_1$$
$$+ \left(\frac{D_{37}}{a} + G_{13}\right)\phi' = \rho_{13}\ddot{v}_0 - \rho_{35}\ddot{w}'_0 + \rho_{23}\ddot{v}_1 - \rho_{34}\ddot{u}'_0 + \rho_{33}\ddot{u}_1 \tag{12e}$$

$$\frac{D_{17}}{a}v'_0 + \frac{D_{67}}{a}w_0 - \frac{D_{47}}{a}w''_0 + \left(\frac{D_{57}}{a} - \bar{G}_{12}\right)v'_1 - \left(\frac{D_{27}}{a} + \frac{G_{23}}{a}\right)u''_0$$
$$+ \left(\frac{D_{37}}{a} + G_{13}\right)u'_1 + D_{77}\frac{\phi}{a^2} - (G_{33} + \bar{G}_{22})\phi''$$
$$= m_\theta - \bar{\bar{\rho}}_{12}\ddot{w}_0 + \bar{\rho}_{12}\ddot{u}_0 - (\bar{\rho}_{22} + \bar{\bar{\rho}}_{22})\ddot{\phi}. \tag{12f}$$

In eqns (12) D_{ij}, G_{ij}, $\bar{G}_{ij}$, etc. are given in Appendix B. This completes the formulation of the governing equations for generalized treatment of the curved beams of constant curvature.

It is to be noted here that for homogeneous and symmetrical cross-sections ($D_{12} = D_{13} = D_{17} = D_{24} = D_{25} = D_{26} = D_{34} = D_{35} = D_{36} = D_{47} = D_{57} = D_{67} = G_{12} = \rho_{13} = \rho_{14} = \rho_{23} = \rho_{24} = \rho_{35} = \rho_{45} = \rho_{12} = 0$) eqns (12) uncouple into two sets (three equations in each set). The first set (eqns (12a)–(12c)) involves the displacement parameters (v_0, w_0, v_1) corresponding to in-plane motions and the second set (eqns (12d)–(12f)) involves the displacement parameters (u_0, u_1, ϕ) corresponding to out-of-plane motion of the ring. This is so even in the case of laminated cross-sections when the laminate direction corresponds to the z-direction (Fig. 1(b)).

However, in the case of a laminated cross-section as shown in Fig. 1(c) eqns (12) uncouple into two sets only when the laminate arrangement is symmetrical with respect to the z-axis. For an arbitrary arrangement of laminates as in Fig. 1(c) eqns (12) have to be solved simultaneously. Also, we note that such coupling of in-plane and out-of-plane motions does not occur in the case of straight beams ($1/a = 0$). For the case of straight beams eqns (12) uncouple into four sets: (i) eqn (12a) involving only v_0; (ii) eqns (12b) and (12c) involving only w_0 and v_1; (iii) eqns (12d) and (12e) involving only u_0 and u_1; (iv) eqn (12f) involving only ϕ.

The equations derived above are general in the sense that any curved beam problem can be addressed. It is difficult to obtain the closed-form solutions to arbitrarily supported curved beams. However, a freely supported ring renders possibility of the closed-form solution to equilibrium equations (12). Hence, it is intended to solve the same problem to evaluate the present theory using the following modal solution:

$$(v_0, v_1, u_1) = (A, C, E) \sin (n\theta) \cos \Omega t$$
$$(w_0, u_0, \phi) = (B, D, F) \cos (n\theta) \cos \Omega t. \qquad (13)$$

Solutions (13) are substituted into eqns (12), in the absence of externally applied loads ($q_z = q_x = m_\theta = 0$), to obtain the following eigenvalue problem:

$$Kp = \Omega^2 Mp \qquad (14)$$

where K and M are 6×6 symmetric positive definite matrices and $p^t = \{A\ B\ C\ D\ E\ F\}$. The coefficients of these matrices are given in Appendix C.

Thus, it is clear that for given ring dimensions and for a given value of $n \geqslant 2$ we are always solving for six frequencies and the associated modes. These six modes comprise of two flexural modes, two shear modes, one torsional mode, and one extensional mode. We understand here, for an uncoupled problem, that out of six frequencies three correspond to in-plane motion and three others correspond to out-of-plane motion. However, in the case of the coupled problem three frequencies can still be identified as corresponding to predominantly in-plane motion and three others as corresponding to predominantly out-of-plane motion.

DISCUSSION OF NUMERICAL RESULTS

To evaluate the applicability of the present theory some results were obtained for rectangular steel rings. These results are presented in Tables 1 and 2. Table 1 shows the comparison of results from various theories for the in-plane flexural vibration mode. It is evident that the present theory predicts, in most cases, very accurate frequencies as compared to other thin and thick ring theories. It may be observed from Table 2 that the out-of-plane flexural frequencies predicted by the present formulation are very accurate as compared to other theories, except in a few cases for $n = 2$.

Table 1. In-plane flexural frequencies (Hz) of rectangular cross-section ring ($E = 207$ GPa; $E/G = 2.58$; $\rho = 7833$ kg m^{-3})

	Mode No.	Exp. Ref. [13]	Ref. [2] (% error)	Present (% error)	Ref. [6] (% error)	Ref. [4] (% error)	Classical (% error)
	2	7635	1.7	0.46	−3.6	−5.0	10
	3	19060	0.5	1.35	−3.2	0.3	24
$a = 36.3$ mm	4	32150	0.1	2.02	−3.0	1.2	41
$b/a = 0.47934$	5	46050	−1.02	1.00	−3.1	1.2	59
$h/a = 0.11047$	6	60400	—	−0.34	−3.3	0.8	78
	7	74200	—	−0.04	−2.7	1.1	100
	8	88000	—	0.32	−2.2	1.3	122
	2	12070	0.9	3.68	−6.7	−4.2	20
	3	28650	−3.0	1.38	−8.8	−5.5	44
$a = 32.6$ mm	4	44750	−3.3	−0.81	−6.6	−3.3	76
$b/a = 0.7546$	5	60200	−3.8	−0.15	−4.4	−1.6	112
$h/a = 0.12301$	6	73900	—	2.9	−1.0	1.5	153
	7	86300	—	6.7	2.7	4.7	198
	8	97950	—	10.6	6.0	7.8	246

% Error = (Experimental value/Theoretical value − 1) × 100.

Table 2. Out-of-plane flexural frequencies (Hz) of square cross-section ring ($E = 207$ GPa; $E/G = 2.58$; $\rho = 7833$ kg m^{-3})

	Mode No.	Exp. Ref. [13]	Present value	Present % error	Ref. [9] value	Ref. [9] % error	Ref. [8] value	Ref. [8] % error
	2	2605	2564	−1.61	2575	−1.15	2608	0.12
	3	7330	7223	−1.47	7254	−1.04	7359	0.40
$a = 41.26$ mm	4	13750	13522	−1.66	13576	−1.27	13784	0.25
$b/a = 0.18126$	5	21450	21137	−1.46	21213	−1.10	21548	0.46
$h/a = 0.18126$	6	30400	29798	−1.98	29889	−1.68	30372	−0.09
	7	40050	39273	−1.94	39373	−1.69	40024	−0.06
	8	50450	49386	−2.11	49479	−1.92	50315	−0.27
	2	6620	6437	−2.77	6491	−1.95	6788	2.54
	3	16790	16347	−2.64	16520	−1.61	17430	3.81
$a = 36.56$ mm	4	28700	28017	−2.38	28138	−1.96	29857	4.03
$b/a = 0.46184$	5	41200	40500	−1.70	40448	−1.83	43121	4.66
$h/a = 0.46184$	6	53950	53373	−1.07	53023	−1.72	56767	5.22
	7	66100	65776	−0.49	65660	−0.67	70567	6.76
	8	79200	78804	−0.50	78262	−1.18	84403	6.57
	2	10000	9999	−3.84	9710	−2.90	10758	7.58
$a = 32.74$ mm	3	23100	22368	−3.17	22498	−2.61	25413	10.01
$b/a = 0.74914$	4	36700	35918	−2.13	35911	−2.15	41072	11.91
$h/a = 0.74914$	5	49900	48982	−1.84	49320	−1.16	56935	14.10
	6	62650	62888	0.38	62569	−0.13	72747	16.12
	2	13485	12591	−6.63	12649	−6.20	15384	14.08
$a = 28.76$ mm	3	28100	27192	−3.23	26945	−4.11	33833	20.40
$b/a = 1.12989$	4	41600	40614	−2.37	41005	−1.43	52458	26.10
$h/a = 1.12989$	5	54500	55045	1.00	54692	0.35	70798	29.90
	6	67800	68709	1.34	68079	0.41	88792	30.96

% Error = (Experimental value/Theoretical value − 1) × 100.

Having evaluated the validity of the present theory we proceed to solve the coupled vibration problem of a laminated ring. As stated earlier coupled in-plane and out-of-plane vibrations occur only when the ring cross-section is asymmetrically laminated in the x-direction (Fig. 1(c)). Hence, numerical results were obtained for a two-layered (0/90 orientation) ring of varied geometrical parameters. Typical elastic properties chosen for a lamina correspond to the following:

$$E_L/E_T = 10; \quad E_L/G_{LT} = E_L/G_{Lz} = 40; \quad \nu_{LT} = 0.25.$$

The geometric parameters are shown in the respective tables.

To illustrate the effect of coupling eqn (14) was solved to obtain six coupled vibration frequencies whereas, purely in-plane frequencies were obtained by solving first three of the

Table 3. Frequencies (Ω/Ω_0 of rectangular cross-section composite ring for various h/b ratios ($\Omega_0 = \sqrt{(E_L/a^2\rho)}$; $a/b = 2$; $n = 2$; $h_1/h_2 = 1$)

	In-plane motion			Out-of-plane motion		
h/b	Flexure	Extension	Shear	Flexure	Torsion	Shear
1/2	0.17002	1.32208	1.92583	0.07591	0.45348	2.65509
	0.17084	1.46175	2.03598	0.08582	0.52571	2.48026
1	0.17044	1.04691	1.78621	0.10083	0.48073	2.31664
	0.17084	1.46175	2.03598	0.11009	0.51396	1.82655
3/2	0.17135	0.93849	1.76281	0.10474	0.49273	2.30886
	0.17084	1.46175	2.03598	0.11478	0.50326	1.73915
2	0.17256	0.91345	1.76505	0.10512	0.48886	2.34231
	0.17084	1.46175	2.03598	0.11796	0.49252	1.77438

Table 4. Frequencies (Ω/Ω_0) of rectangular cross-section composite ring for various h_1/h_2 ratios ($\Omega_0 = \sqrt{(E_L/a^2\rho)}$; $a/b = 2$; $n = 2$; $h/b = 1$)

	In-plane motion			Out-of-plane motion		
h_1/h_2	Flexure	Extension	Shear	Flexure	Torsion	Shear
1	0.17044	1.04691	1.78621	0.10083	0.48073	2.31664
	0.17084	1.46175	2.03598	0.11009	0.51396	1.82655
2	0.17832	1.31031	1.90698	0.10475	0.49853	2.40562
	0.17844	1.63254	2.23900	0.10975	0.51564	1.85574
3	0.18148	1.49150	1.96639	0.10685	0.50727	2.44369
	0.18152	1.71060	2.33464	0.10990	0.51784	1.91696
4	0.18319	1.60612	2.00737	0.10815	0.51298	2.46774
	0.18321	1.75554	2.39036	0.11019	0.52021	1.97363

Table 5. Frequencies (Ω/Ω_0) of rectangular cross-section composite ring for various n values ($\Omega_0 = \sqrt{(E_L/a^2\rho)}$; $a/b = 2$; $h/b = 1$; $h_1/h_2 = 1$)

	In-plane motion			Out-of-plane motion		
n	Flexure	Extension	Shear	Flexure	Torsion	Shear
2	0.17044	1.04691	1.78621	0.10083	0.48073	2.31664
	0.17084	1.46175	2.03598	0.11009	0.51396	1.82655
3	0.36618	1.34616	2.42275	0.25571	0.61144	3.21357
	0.36727	2.05224	2.79153	0.26822	0.62463	2.44559
4	0.57265	1.66927	3.10218	0.42198	0.76670	4.15214
	0.57392	2.66817	3.58872	0.43328	0.77249	3.10357
5	0.79513	2.00544	3.80148	0.59116	0.93514	5.11067
	0.79653	3.29503	4.40595	0.59933	0.93752	3.78233

six equilibrium equations, eqns (12a)–(12c), ignoring the terms associated with u_0, u_1, ϕ displacement parameters. Similarly purely out-of-plane frequencies were obtained by solving the last three of the six equilibrium equations, eqns (12d)–(12f), ignoring the terms associated with v_0, w_0, v_1 displacement parameters. The results are tabulated in Tables 3–5. In these tables, numbers in the first row correspond to the coupled problem and those in the second row correspond to the uncoupled problem.

It may be observed from these tables that the coupling does not affect the in-plane flexural frequencies, whereas those of the extensional mode are severely affected (error ranging to 65% in some cases). To a lesser extent, as compared to the extensional mode, other frequencies are also affected by the coupling action. Even for the present case of $E_L/E_T = 10$ the maximum error in the out-of-plane frequency is 13% and corresponding errors in the frequencies of shear modes and the torsional mode are about 13 and 16%. These errors increase as lamina properties become highly anisotropic. There are composites for which E_L/E_T becomes as large as 20. For such composite rings these errors will be still higher.

It is a well established fact that, for two-layered composites, the h_1/h_2 ratio is the one which makes the cross-section more asymmetrical (as regard to elastic properties) or less asymmetrical depending on its value. As the h_1/h_2 ratio increases the cross-section becomes less asymmetrical and hence the coupling effects gradually diminish. This is clearly observable in Table 4 in which uncoupled frequencies gradually approach those of coupled frequencies as the h_1/h_2 ratio increases. In general, the effect of coupling on out-of-plane flexural and torsional frequencies decreases as the out-of-plane thickness (h) increases, as the mode number (n) increases, and as the thickness to radius (b/a) ratio increases. Whereas, the effect of coupling on the extensional and shear modes increases for increasing values of h, n, and b/a.

CONCLUSIONS

The theory presented above considers parabolic shear strain variations and involves six centre-line displacement parameters to be determined to obtain the complete solution of the problem. The careful selection of basic displacement components makes it possible to obtain the classical and the Timoshenko-type shear deformation theories as special cases of the present theory. Accurate prediction of frequencies for steel rings ensures the validity of the present formulation. It is interesting to note that coupling exists between in-plane and out-of-plane motions for some composite rings. Thus, such structures require the complete analysis involving all six equilibrium equations. The numerical results presented in Tables 3–5 for two-layered rings demonstrate the errors involved in frequency predictions when a separate analysis is carried out for purely in-plane and purely out-of-plane motions. The errors are high enough to demand the complete vibration analysis to be performed.

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APPENDIX A

Expressions for stress resultants in eqn (10)

$$\begin{bmatrix} N_\theta \\ M_z \\ \bar{M}_z \\ M_x \\ \bar{M}_x \end{bmatrix} = \begin{bmatrix} D_{11} & D_{16} & D_{14} & D_{15} & D_{12} & D_{13} & D_{17} \\ D_{12} & D_{26} & D_{24} & D_{25} & D_{22} & D_{23} & D_{27} \\ D_{13} & D_{36} & D_{34} & D_{35} & D_{23} & D_{33} & D_{37} \\ D_{14} & D_{46} & D_{44} & D_{45} & D_{24} & D_{34} & D_{47} \\ D_{15} & D_{56} & D_{45} & D_{55} & D_{25} & D_{35} & D_{57} \end{bmatrix} \begin{bmatrix} v_0' \\ w_0/a \\ -w_0'' \\ v_1' \\ -u_0'' \\ u_1' \\ \phi/a \end{bmatrix}$$

$$\begin{bmatrix} \bar{Q}_x \\ \bar{Q}_z \\ M_\theta \end{bmatrix} = \begin{bmatrix} 0 & -G_{12} & -G_{11} & -G_{13} \\ \bar{G}_{11} & 0 & 0 & G_{12} \\ \bar{G}_{12} & G_{23} & -\bar{G}_{13} & G_{33}+\bar{G}_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ u_0'/a \\ u_1 \\ \phi' \end{bmatrix}.$$

APPENDIX B

Definitions of D_{ij}, G_{ij}, $\bar{G}_{ij}$, ρ_{ij}, $\bar{\rho}_{ij}$, $\bar{\bar{\rho}}_{ij}$ in eqns (12)

$$\mathbf{D} = \int_A C_{11} \begin{bmatrix} 1 & x & f_x & z\bar{a} & f_z & \bar{a} & x\bar{a} \\ & x^2 & xf_x & xz\bar{a} & xf_z & x\bar{a} & x^2\bar{a} \\ & & f_x^2 & zf_x\bar{a} & f_xf_z & f_x\bar{a} & xf_x\bar{a} \\ & & & z^2\bar{a}^2 & zf_z\bar{a} & z\bar{a}^2 & xz\bar{a}^2 \\ \text{Sym.} & & & & f_z^2 & f_z\bar{a} & xf_z\bar{a} \\ & & & & & \bar{a}^2 & x\bar{a}^2 \\ & & & & & & x^2\bar{a}^2 \end{bmatrix} \frac{\mathrm{d}x\,\mathrm{d}z}{\bar{a}}$$

$$G_{1i} = G_{i1} = \int C_{66}\alpha_i\alpha_1 \frac{\mathrm{d}x\,\mathrm{d}z}{\bar{a}} \quad (i = 1,2,3)$$

$$G_{ij} = \frac{12C_k}{hb^3}\int C_{66}\alpha_i\alpha_j \frac{\mathrm{d}x\,\mathrm{d}z}{\bar{a}} \quad (i = 2,3;\ j = 2,3)$$

$$\bar{G}_{1i} = \bar{G}_{i1} = \int C_{44}\bar{\alpha}_i\bar{\alpha}_1 \frac{\mathrm{d}x\,\mathrm{d}z}{\bar{a}} \quad (i = 1,2)$$

$$\bar{G}_{22} = \frac{12C_k}{bh^3}\int C_{44}\bar{\alpha}_2^2 \frac{\mathrm{d}x\,\mathrm{d}z}{\bar{a}}$$

$$\alpha_1 = f_z^*/\bar{a}; \quad \alpha_2 = z\left(\frac{2a+z}{a+z}\right); \quad \alpha_3 = z\bar{a};$$

$$\bar{\alpha}_1 = f_z^*/\bar{a}; \quad \bar{\alpha}_2 = x\bar{a}; \quad \bar{a} = \left(\frac{a}{a+z}\right)$$

$$\rho = \int_A \rho \begin{bmatrix} 1 & f_z & f_x & x & z\bar{a} \\ & f_z^2 & f_xf_z & xf_x & zf_z\bar{a} \\ & & f_x^2 & xf_x & zf_x\bar{a} \\ \text{Sym.} & & & x^2 & xz\bar{a} \\ & & & & z^2\bar{a}^2 \end{bmatrix} \frac{\mathrm{d}x\,\mathrm{d}z}{\bar{a}^3}$$

$$\bar{\rho} = \int_A \rho \begin{bmatrix} 1 & z \\ z & z^2 \end{bmatrix} \frac{\mathrm{d}x\,\mathrm{d}z}{\bar{a}}; \quad \bar{\bar{\rho}} = \int_A \rho \begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix} \frac{\mathrm{d}x\,\mathrm{d}z}{\bar{a}}.$$

APPENDIX C

Coefficients of K and M matrices in eqn (14)

$$K_{11} = D_{11}\left(\frac{n}{a}\right)^2, \quad K_{12} = D_{16}\frac{n}{a^2} + D_{14}\left(\frac{n}{a}\right)^3, \quad K_{13} = D_{15}\left(\frac{n}{a}\right)^2$$

$$K_{14} = D_{12}\left(\frac{n}{a}\right)^3, \quad K_{15} = D_{13}\left(\frac{n}{a}\right)^2, \quad K_{16} = D_{17}\left(\frac{n}{a}\right)\frac{1}{a}$$

$$K_{22} = D_{44}\left(\frac{n}{a}\right)^4 + 2D_{46}\left(\frac{n}{a}\right)^2\frac{1}{a} + \frac{D_{66}}{a^2}, \qquad K_{23} = \frac{D_{56}}{a}\left(\frac{n}{a}\right) + D_{45}\left(\frac{n}{a}\right)^3$$

$$K_{24} = D_{24}\left(\frac{n}{a}\right)^4 + \frac{D_{26}}{a}\left(\frac{n}{a}\right)^2, \qquad K_{25} = D_{36}\left(\frac{n}{a}\right)\frac{1}{a} + D_{34}\left(\frac{n}{a}\right)^3$$

$$K_{26} = \frac{D_{67}}{a^2} + \frac{D_{47}}{a}\left(\frac{n}{a}\right)^2, \qquad K_{33} = D_{55}\left(\frac{n}{a}\right) + \bar{G}_{11}$$

$$K_{34} = D_{25}\left(\frac{n}{a}\right)^3, \qquad K_{35} = D_{35}\left(\frac{n}{a}\right)^2, \qquad K_{36} = \left(\frac{D_{57}}{a} - \bar{G}_{12}\right)\frac{n}{a}$$

$$K_{44} = D_{22}\left(\frac{n}{a}\right)^4 + \frac{G_{22}}{a^2}\left(\frac{n}{a}\right)^2, \qquad K_{45} = \frac{G_{12}}{a}\left(\frac{n}{a}\right) + D_{23}\left(\frac{n}{a}\right)^3$$

$$K_{46} = \left(\frac{D_{27}}{a} + \frac{G_{23}}{a}\right)\left(\frac{n}{a}\right)^3, \qquad K_{55} = D_{33}\left(\frac{n}{a}\right)^2 + G_{11}$$

$$K_{56} = \left(\frac{D_{37}}{a} + G_{13}\right)\left(\frac{n}{a}\right), \qquad K_{66} = D_{77}\frac{1}{a^2} + (G_{33} + \bar{G}_{22})\left(\frac{n}{a}\right)^2$$

$$M_{11} = \rho_{11}, \qquad M_{12} = \rho_{15}\left(\frac{n}{a}\right), \qquad M_{13} = \rho_{12}, \qquad M_{14} = \rho_{14}\left(\frac{n}{a}\right)$$

$$M_{15} = \rho_{13}, \qquad M_{16} = 0, \qquad M_{22} = \bar{\bar{\rho}}_{11} + \rho_{55}\left(\frac{n}{a}\right)^2, \qquad M_{23} = \rho_{25}\left(\frac{n}{a}\right)$$

$$M_{24} = \rho_{45}\left(\frac{n}{a}\right)^2, \qquad M_{25} = \rho_{35}\left(\frac{n}{a}\right), \qquad M_{26} = \bar{\bar{\rho}}_{12}, \qquad M_{33} = \rho_{22}$$

$$M_{34} = \rho_{24}\left(\frac{n}{a}\right), \qquad M_{35} = \rho_{23}\left(\frac{n}{a}\right), \qquad M_{36} = 0, \qquad M_{44} = \bar{\rho}_{11} + \rho_{44}\left(\frac{n}{a}\right)^2$$

$$M_{45} = \rho_{34}\left(\frac{n}{a}\right), \qquad M_{46} = -\bar{\rho}_{12}, \qquad M_{55} = \rho_{33}, \qquad M_{56} = 0$$

$$M_{66} = (\bar{\rho}_{22} + \bar{\bar{\rho}}_{22}).$$

APPENDIX D: NOMENCLATURE

a, b, h	mean radius, radial width, out-of-plane width of curved beam
C_k	torsional constant
$D_{ij}, G_{ij}, \bar{G}_{ij}$	integrated stiffness coefficients (Appendix B)
E, G	elastic modulus and shear modulus for homogeneous material
E_L, E_T	elastic moduli of a typical layer of composite ring
G_{LT}, G_{Lz}	shear moduli of a typical layer of composite ring
ν_{LT}	Poisson's ratio for a typical layer in composite ring
$N_\theta, M_z, \bar{M}_z, M_x$, $\bar{M}_x, \bar{Q}_x, \bar{Q}_z, M_\theta$, $N_\theta^*, M_z^*, Q_x, Q_z$	stress resultants as defined in eqns (10)
n	mode number
m_θ	applied torsional moment per unit length on centre-line
q_x, q_z	applied loads in x- and z-direction per unit length
t	time coordinate
u, v, w	displacements in x-, θ-, and z-direction
u_0, v_0, w_0	displacements of a point on the centre-line
u_1, v_1, ϕ	rotations of a point on the centre-line
x, θ, z	cylindrical coordinate system
$\varepsilon_x, \varepsilon_z, \varepsilon_\theta$	normal strains
$\gamma_{\theta x}, \gamma_{\theta z}, \gamma_{xz}$	shear strains
σ_θ	normal stress in θ-direction
$\tau_{\theta x}, \tau_{\theta z}$	shear stresses on θ-plane
ρ	density of the material
$\rho_{ij}, \bar{\rho}_{ij}, \bar{\bar{\rho}}_{ij}$	integrated inertia coefficients (Appendix B)
Ω	radian frequency.